

Fluctuations in the irreversible decay of turbulent energy

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A fluctuation law of the energy in freely decaying, homogeneous, and isotropic turbulence is derived within standard closure hypotheses for three-dimensional incompressible flow. In particular, a fluctuation-dissipation relation is derived which relates the strength of a stochastic backscatter term in the energy decay equation to the mean of the energy dissipation rate. The theory is based on the so-called “effective action” of the energy history and illustrates a Rayleigh-Ritz method recently developed to evaluate the effective action approximately within probability density-function (PDF) closures. These effective actions generalize the Onsager-Machlup action of nonequilibrium statistical mechanics to turbulent flow. They yield detailed, concrete predictions for fluctuations, such as multitime correlation functions of arbitrary order, which cannot be obtained by direct PDF methods. They also characterize the mean histories by a variational principle. [S1063-651X(97)00811-8]

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I. INTRODUCTION

We consider here the problem of fluctuations of the energy in high Reynolds number turbulence decay. The mean energy decay in homogeneous and isotropic turbulence has been the subject of many classic investigations. A rather thorough review is contained in [1], Sec. 16. von Kármán and Howarth [2] derived a power decay law for the mean energy, $K_*(t) \propto (t-t_0)^{-n}$, by means of a hypothesis of complete self-preservation of the spectrum. The fundamental paper of Kolmogorov [3] rederived that result, with a precise prediction for the exponent, $n = \frac{10}{7}$. Kolmogorov’s original argument assumed, however, the conservation of the Loitsyansky invariant, which was later called into question by Proudman and Reid [4]. Nevertheless, Kolmogorov’s basic argument may still be carried through under a weaker hypothesis, a “principle of permanence of the large eddies.” This now-standard theory has been discussed in several books and reviews: [5–7]. According to this theory, the decay exponent n is dependent on the initial data, through the power of the low-wave-number part of the spectrum.

Our interest here is in the fluctuations of the energy history during the decay, including joint multitime statistics. The main results have been briefly announced elsewhere [8]. Our analysis is based on a general approach to fluctuations in irreversible processes, proposed by Onsager [9] and developed in detail by Onsager and Machlup [10]. In this method, an “action functional” is employed which measures directly the probability of observing a given history as a fluctuation event. In particular, the most probable history minimizes this action functional. In systems close to thermal equilibrium, there is a standard fluctuation-dissipation relation for molecular noise, so that the Onsager-Machlup action has there the physical interpretation of a “dissipation function.” Onsager’s variational principle reduces near equilibrium to a “principle of least dissipation,” generalizing the well-known hydrodynamic principle of Rayleigh.

In its original form, however, Onsager’s principle was restricted to weakly noisy systems and could not be applied

to turbulence, where fluctuations are large. Recently we have proposed a generalization which applies as well to strongly noisy systems [11,12]. The variational functionals in this theory, or “effective actions,” have experimental consequences for turbulence fluctuations and are subject to realizability conditions which arise from positivity of the underlying statistical distributions. For each random variable $Z(t)$ in the flow (where Z may represent a velocity at a chosen point, a pressure, a turbulent energy, etc.) there is a corresponding effective action $\Gamma[z]$, which is a functional of the whole time history $\{z(t):t_0 \leq t < +\infty\}$ of the variable. The realizability conditions on this action function are (i) that it be nonnegative, $\Gamma[z] \geq 0$, (ii) that it have the ensemble mean $\bar{z}(t)$ as its unique minimum $\Gamma[\bar{z}] = 0$, and (iii) that it be convex, $\lambda \Gamma[z_1] + (1-\lambda)\Gamma[z_2] \geq \Gamma[\lambda z_1 + (1-\lambda)z_2]$, $0 < \lambda < 1$. As a consequence, the mean value $\bar{z}(t)$ is characterized by a *principle of least effective action*. Like Onsager’s action, this functional directly measures the probability of fluctuations of the sample histories away from the mean history. The effective action also serves as a generating functional for (irreducible) multitime correlation functions of the considered random variable.

To make the effective action into a practical working tool, efficient and economical approximation procedures are required to calculate it. In [11,12] we have demonstrated one such scheme, a Rayleigh-Ritz variational method inspired by the similar ones already extensively used in quantum theory. This variational method is designed to be used in conjunction with probability density-function (PDF) closures, such as mapping closures [13,14], generalized Langevin models [15,16], etc. Any reasonable guess of the turbulence statistics may be input into the variational method to yield approximations of the effective actions. By this means, predictions are obtained for multitime statistics which are not obtainable by direct PDF methods. The additional information about fluctuations has been found to be very useful in evaluating the reliability of PDF closures for practical modeling purposes [8].

The contents of this paper are as follows. In Sec. II we very briefly review the standard theory of the mean energy

decay in high Reynolds number turbulence and, in particular, we cast it in the form of a PDF-based moment closure. In Sec. III we evaluate the effective action within the standard theoretical hypotheses, by means of the Rayleigh-Ritz algorithm. The realizability of the approximate effective action is verified in the small-fluctuation regime by means of a Langevin dynamics for the turbulent energy and a fluctuation-dissipation relation is derived for the strength of the stochastic noise term. In Sec. IV we discuss some of the testable consequences of the theory. In particular, the prediction for the two-time correlation of the turbulent energy is given. Also, the direct empirical significance of the effective action is discussed, in terms of fluctuations in N -sample ensemble averages.

II. REVIEW OF THEORY FOR THE MEAN DECAY

We outline here the standard theory of mean energy decay in a freely decaying homogeneous and isotropic turbulence with random initial data at high Reynolds number, following the reviews in [5–7]. For convenience, we assume a model energy spectrum

$$E(k,t) = \begin{cases} Ak^m, & k \leq k_L(t) \\ \alpha \varepsilon^{2/3}(t) k^{-5/3}, & k_L(t) \leq k \leq k_d(t) \\ 0, & k \geq k_d(t) \end{cases} \quad (2.1)$$

which has been adopted in some previous studies [17,6]. Such a spectrum may certainly be taken at time $t=t_0$ for the initial velocity statistics. We are assuming as well that there is a permanent form of the spectrum, according to which the spectral shape is unchanged in time except through its dependence on the parameters $\varepsilon(t)$, $k_L(t)$, and $k_d(t)$. Note that the spectrum is not self-preserving, or self-similar, in the usual sense discussed in [1], which would imply that it have the form $E(k,t) = \alpha \varepsilon^{2/3}(t) k^{-5/3} f(k\ell(t))$ for some length scale $\ell(t)$. In fact, the spectrum contains two distinct length scales, the integral or outer scale $L(t) = k_L^{-1}(t)$ and the dissipation or inner scale $\eta(t) = k_d^{-1}(t)$. Certain features of the above model are crude caricatures of reality. For example, the spectrum should not vanish for $k > k_d(t)$ at any time $t > t_0$, even if it did so initially. However, the spectrum should always show some rapid exponential decay in the far dissipation range. It may be easily checked that such refinements do not change any of the results below. The important assumption has to do with the low-wave-number part of the spectrum. It was found by Proudman and Reid from the quasinnormal closure [4] that there is a backscatter term $\sim k^4$ in the energy transfer $T(k,t)$. Hence, as long as $m < 4$ one expects that there is a permanence of the low-wave-number spectrum, in the sense that the power law k^m and its coefficient A remain unchanged in time. On the other hand, if $m > 4$ initially, then it is expected that the spectrum with $m=4$ will be established at positive times and, if $m=4$ initially, then the low-wave-number spectrum will remain of the same form with a time-dependent coefficient $A(t)$. Here we always consider $m < 4$, so that the ‘‘permanence of large eddies’’ should hold. For finiteness of the total energy, $m > -1$ must also be imposed and, in fact, we usually take

$m > 0$ so that the spectrum decreases asymptotically at very low wave numbers. For grid-generated turbulence it has been inferred that $m \approx 2$ [17].

The mean decay law can be derived very simply from these hypotheses. One relation $k_L(t) = [(\alpha/A)\varepsilon^{2/3}(t)]^{3/(3m+5)}$ is imposed on the spectral parameters by requiring continuity at $k = k_L(t)$. An additional constraint is obtained at high Reynolds number by evaluating the dissipation rate

$$\varepsilon(t) = 2\nu \int_0^\infty k^2 E(k,t) dk, \quad (2.2)$$

which, for $k_L(t) \ll k_d(t)$, leads to $k_d(t) = (2/3\alpha\nu)^{3/4} \varepsilon^{1/4}(t)$. Only one independent parameter is left, which may be taken to be the integral $E(t) = \int_0^\infty dk E(k,t)$ representing mean energy at time t . For the above form of the spectrum it is not hard to show that at high Reynolds number, when $k_L(t) \ll k_d(t)$, the dissipation is given as

$$\varepsilon(t) = \Lambda_m E^p(t), \quad (2.3)$$

with $\Lambda_m^{-1} = \alpha^{3/2} [1/(m+1) + \frac{3}{2}]^{(3m+5)/(2m+2)} A^{1/(m+1)}$ and $p = (3m+5)/(2m+2)$. Thus, employing the Navier-Stokes equation via its energy balance, one obtains the closed moment equation

$$\dot{E}(t) = -\Lambda_m E^p(t). \quad (2.4)$$

This generalizes the decay equation (29) in the paper [3] of Kolmogorov. Its solution with initial condition $E(t_0) = K_0$ gives a prediction for the energy decay law, in the form

$$K_*(t) = K_0 \left(\frac{t - t_0^*}{\Delta t} \right)^{-n}, \quad (2.5)$$

with $n = (2m+2)/(m+3)$. Here $\Delta t \equiv [\Lambda_m(p-1)K_0^{p-1}]^{-1}$ is a constant with units of time, determined by the initial mean energy K_0 , and $t_0^* \equiv t_0 - \Delta t$ is a ‘‘virtual time origin.’’

This simple theory may be cast into the form of a PDF closure by assuming as an *Ansatz* at all times $t \geq t_0$ a Gaussian random velocity field with zero mean and with spectrum $E(k,t)$ given by Eq. (2.1). The assumption of Gaussian statistics was not used in previous works. It is not necessary here either, but it makes simpler the analytical labor in applying the Rayleigh-Ritz algorithm. In fact, the Gaussian *Ansatz* will only be used to evaluate averages of one-point velocity moments, which it is known have statistics in actual fact close to Gaussian. See [18], Chap. VIII. It will be shown later that the use of a non-Gaussian *Ansatz* would change only a constant in the final results. The model spectrum contains one free parameter, which may be taken to be the energy per mass $E(t)$ in the *Ansatz*, or, what is the same, its mean value of the quadratic velocity-moment functional $\hat{K}(\mathbf{r}; \mathbf{v}) = \frac{1}{2} v^2(\mathbf{r})$ at some chosen space point \mathbf{r} . By statistical homogeneity, the mean value is independent of this choice. The time dependence of $E(t)$ is then determined by projecting the Navier-Stokes dynamics onto this single moment function:

$$\dot{E}(t) = \langle \mathcal{L}^\dagger \hat{K}(\mathbf{r}) \rangle_{E(t)}, \quad (2.6)$$

where

$$\mathcal{L} = - \sum_{i=1}^3 \int d^3 \mathbf{r} \frac{\delta}{\delta v_i(\mathbf{r})} (\{ -[\mathbf{v}(\mathbf{r}) \cdot \nabla] v_i(\mathbf{r}) - \nabla_i p(\mathbf{r}) + \nu \Delta v_i(\mathbf{r}) \} \cdots) \quad (2.7)$$

is the Liouville operator which generates the evolution of PDF's for the Navier-Stokes dynamics, \mathcal{L}^\dagger is the adjoint operator which generates the evolution of observables, and $\langle \cdots \rangle_{E(t)}$ denotes average with respect to the model Gaussian velocity with energy $E(t)$. It is easy to see that this prescription to determine the time dependence leads to

$$\dot{E}(t) = -\varepsilon(t), \quad (2.8)$$

which, using Eq. (2.3), is clearly equivalent to the one above. However, putting the analysis into this form allows us to apply the Rayleigh-Ritz method of [11,12] to evaluate the effective actions.

III. CALCULATION OF THE ACTION

We calculate here the effective action $\Gamma[K]$ of the energy history $\hat{K}(\mathbf{r}, t; \mathbf{v}) = \frac{1}{2} v^2(\mathbf{r}, t)$. For each time t , $\hat{K}(\mathbf{r}, t)$ is a functional on phase space via its dependence on the random initial data $\mathbf{v}(\mathbf{r})$ at $t=t_0$ of the Navier-Stokes solution $\mathbf{v}(\mathbf{r}, t)$. $K(t)$ is a possible value of this random variable, i.e., a numerical time history. According to the theorem established in [11,12], the effective action is characterized as the stationary point of the ‘‘nonequilibrium action functional’’

$$\Gamma[\hat{A}, \hat{\rho}] = \int_{t_0}^{\infty} dt \langle \hat{A}(t), (\partial_t - \mathcal{L}) \hat{\rho}(t) \rangle \quad (3.1)$$

varied over arbitrary left and right ‘‘trial functionals’’ $\hat{A}[\mathbf{v}; t]$ and $\hat{\rho}[\mathbf{v}; t]$. (In [12] these were denoted Ψ^L , Ψ^R , respectively; the hat is used here to denote functionals on the phase space of velocity fields.) The variations are performed subject to the constraints of unit overlap

$$\langle \hat{A}(t), \hat{\rho}(t) \rangle = 1 \quad (3.2)$$

and fixed mean [of $\hat{K}(\mathbf{r})$ not of $\hat{K}(\mathbf{r}, t)$]

$$\langle \hat{A}(t), \hat{K}(\mathbf{r}) \hat{\rho}(t) \rangle = K(t), \quad (3.3)$$

with the initial condition

$$\hat{\rho}[\mathbf{v}; t_0] = \hat{P}_0[\mathbf{v}], \quad (3.4)$$

where \hat{P}_0 is the initial Gaussian distribution at $t=t_0$, and with the final condition

$$\hat{A}[\mathbf{v}; +\infty] \equiv 1. \quad (3.5)$$

Note that $\langle \hat{A}, \hat{\rho} \rangle = \int \mathcal{D}\mathbf{v} \hat{A}[\mathbf{v}] \hat{\rho}[\mathbf{v}]$. The trial functional $\hat{\rho}(t)$ should be taken to vary over the space of all probability distributions, while $\hat{A}(t)$ is varied over the space of all bounded observables. In this case the constraints become, more simply,

$$\langle \hat{A}(t) \rangle_{\hat{\rho}(t)} = 1 \quad (3.6)$$

and

$$\langle \hat{A}(t) \hat{K} \rangle_{\hat{\rho}(t)} = K(t), \quad (3.7)$$

in which $\langle \cdots \rangle_{\hat{\rho}(t)}$ denotes the average over the distribution $\hat{\rho}(t)$. Likewise,

$$\Gamma[\hat{A}, \hat{\rho}] = - \int_{t_0}^{\infty} dt \langle (\partial_t + \mathcal{L}^\dagger) \hat{A}(t) \rangle_{\hat{\rho}(t)} \quad (3.8)$$

is a generally more useful expression for the nonequilibrium action.

To obtain the exact effective action, trial functionals should be varied over the full spaces. However, within the Gaussian *Ansatz* above, the variation is taken over a restricted class of trial functionals. The right functional is just the Gaussian PDF itself:

$$\hat{\rho}[\mathbf{v}; t] = \frac{1}{\mathcal{N}(t)} \exp \left[- \frac{1}{2} \int d^3 \mathbf{k} \hat{v}_i^*(\mathbf{k}) (E^{-1})_{ij}(\mathbf{k}, t) \hat{v}_j(\mathbf{k}) \right], \quad (3.9)$$

with the isotropic spectral tensor

$$E_{ij}(\mathbf{k}, t) = \frac{E(k, t)}{4\pi k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \quad (3.10)$$

in which $E(k, t)$ is the scalar spectrum of Eq. (2.1) for a variable total energy $E(t)$. $\mathcal{N}(t)$ is the normalization factor guaranteeing total probability equal to unity. The left trial functional within the Gaussian PDF closure is chosen from among arbitrary linear combinations of the moment functional $\hat{K}[\mathbf{r}; \mathbf{v}]$, which appeared in the closure, and the constant functional $\equiv 1$:

$$\hat{A}[\mathbf{v}; t] = \alpha_0(t) 1 + \alpha_1(t) \hat{K}[\mathbf{r}; \mathbf{v}]. \quad (3.11)$$

The variable functions of time, $E(t), \alpha_0(t), \alpha_1(t)$, are the trial parameters of the variational calculation.

However, because of the two constraints, Eqs. (3.6), (3.7), only one of these parameters is independent. We shall take it to be $E(t)$. The unit overlap condition Eq. (3.6) requires that $\alpha_0(t) + \alpha_1(t) E(t) = 1$, or that

$$\hat{A}[\mathbf{v}; t] = 1 + \alpha_1(t) \{ \hat{K}[\mathbf{r}; \mathbf{v}] - E(t) \} \quad (3.12)$$

by eliminating $\alpha_0(t)$. Next $\alpha_1(t)$ may be eliminated by using the condition Eq. (3.7). Since

$$\langle \hat{A}(t) \hat{K}(\mathbf{r}) \rangle_{E(t)} = E(t) + \alpha_1(t) [\langle \hat{K}^2(\mathbf{r}) \rangle_{E(t)} - E^2(t)], \quad (3.13)$$

the constraint equation is obtained from an easily calculated average over the Gaussian ensemble parametrized by $E(t)$.

Using $\langle v_i(\mathbf{r}) v_j(\mathbf{r}) \rangle_{E(t)} = \frac{2}{3} \delta_{ij} E(t)$, this average is found to be

$$\langle \hat{K}^2(\mathbf{r}) \rangle_{E(t)} = \frac{5}{3} E^2(t). \quad (3.14)$$

Note that evaluation of this one-point moment is the only place where Gaussian statistics is employed in the whole calculation. From the imposed condition Eq. (3.7) we then obtain that

$$\alpha_1(t) = \frac{3}{2} E^{-2}(t) [K(t) - E(t)]. \quad (3.15)$$

The action may now be approximated as

$$\begin{aligned} \Gamma_*[K; E] &= - \int_{t_0}^{\infty} dt \langle (\partial_t + \mathcal{L}^\dagger) \hat{A}(t) \rangle_{E(t)} \\ &= \int_{t_0}^{\infty} dt \{ -\dot{\alpha}_1(t) \langle \hat{K} - E(t) \rangle_{E(t)} + \alpha_1(t) \\ &\quad \times [\dot{E}(t) + \langle \hat{\varepsilon} \rangle_{E(t)}] \} \\ &= \int_{t_0}^{\infty} dt \alpha_1(t) [\dot{E}(t) + \Lambda_m E^p(t)] \\ &= \frac{3}{2} \int_{t_0}^{\infty} dt E^{-2}(t) [K(t) - E(t)] \\ &\quad \times [\dot{E}(t) + \Lambda_m E^p(t)], \end{aligned} \quad (3.16)$$

in which $E(t)$ remains as the only trial parameter. We wrote as $\hat{\varepsilon}(\mathbf{r}) = (\nu/2) \sum_{ij} [\partial_i v_j(\mathbf{r}) + \partial_j v_i(\mathbf{r})]^2$ the local energy dissipation rate and noted its average from Eq. (2.3) as $\langle \hat{\varepsilon} \rangle_{E(t)} = \Lambda_m E^p(t)$. By requiring stationarity of the action under variations of $E(t)$, or, $\delta \Gamma_*[K; E] / \delta E(t) = 0$, with fixed $K(t)$, it is straightforward to derive the variational equation

$$\Lambda_m E^p(t) + \dot{K}(t) = (p-2) \Lambda_m [K(t) - E(t)] E^{p-1}(t). \quad (3.17)$$

For any rational value of $p = k/l$, k, l integers, this is a polynomial of degree k in $X = E^{1/l}$: $(p-1) \Lambda_m X^k - (p-2) \Lambda_m K X^{k-1} + \dot{K} = 0$. For a physically allowable energy history, $K(t) > 0$ and $\dot{K}(t) < 0$. Furthermore, $p > 1$ whenever $m > -3$. Thus—independent of the sign of $(p-2)$ —the polynomial has one change of sign in its coefficients for any permissible energy history. Therefore it follows from Descartes' rule of signs that there is exactly one positive root $E(t)$ for each physical choice of $K(t)$, when p is rational. Because these are dense in the real $p > 0$, E is uniquely defined for all permissible K . Substituting that value into the Eq. (3.16) above, we obtain the final form of the approximate effective action

$$\begin{aligned} \Gamma_*[K] &= \frac{3}{2(p-2) \Lambda_m} \\ &\quad \times \int_{t_0}^{\infty} dt \frac{[\dot{K}(t) + \Lambda_m E^p(t)] [\dot{E}(t) + \Lambda_m E^p(t)]}{E^{p+1}(t)} \end{aligned} \quad (3.18)$$

in which the E dependence is eliminated by inserting the root of Eq. (3.17) as described.

It is easy to check that, if the approximate action is evaluated at the predicted closure mean energy $K_*(t)$, then

$\Gamma_*[K_*] = 0$. In fact, using $\dot{K}_*(t) = -\Lambda_m K_*^p(t)$, the corresponding $E_*(t)$ is determined from

$$\Lambda_m [E_*^p(t) - K_*^p(t)] = (p-2) \Lambda_m [K_*(t) - E_*(t)] E_*^{p-1}(t). \quad (3.19)$$

The solution of this equation is

$$E_*(t) = K_*(t) \quad \text{and} \quad \dot{E}_*(t) = -\Lambda_m E_*^p(t). \quad (3.20)$$

Obviously, substituting these values makes the approximate action vanish identically. It can, in fact, be shown that the mean value for *any* closure is a zero of the approximate action evaluated by the Rayleigh-Ritz method within that same closure [11,12]. It may even be shown further that the mean value is always a stationary point of the action, $\delta \Gamma_*[K_*] / \delta K(t) = 0$. However, it need not be a minimum point, as required by the realizability conditions on the effective action.

To examine the issue of realizability here, we consider small perturbations $K(t) = K_*(t) + \delta K(t)$ from the predicted mean. Because the calculation is straightforward but somewhat tedious, we give the details in Appendix A. The final result is that

$$\begin{aligned} \Gamma_*[K] &= \frac{3}{8(p-1) \Lambda_m} \int_{t_0}^{\infty} dt \frac{[\delta \dot{K}(t) + \Lambda_m p K_*^{p-1}(t) \delta K(t)]^2}{K_*^{p+1}(t)} \\ &\quad + O((\delta K)^3). \end{aligned} \quad (3.21)$$

Note again that the coefficient $(p-1)$ in front of the action is > 0 as long as $m > -3$. In fact, $m > -1$ is required to give a finite energy. Thus, for all permissible values of m , the approximate action $\Gamma_*[K]$ satisfies realizability, at least in a small neighborhood of the mean energy history $K_*(t)$. One should be cautioned that satisfaction of realizability is only a consistency check and cannot guarantee correctness of predictions. Indeed, the same calculation as we made above would carry through exactly for the one-dimensional (1D) Burgers equation, since the only property of the nonlinear dynamics that was used was energy conservation. However, not all of the previous results are true for Burgers turbulence. In that case the energy spectrum Eq. (2.1) is not even the correct quasiequilibrium form but, instead, a k^{-2} spectrum will develop [19]. This graphically illustrates that realizability is perfectly compatible with falsity. On the other hand, we expect that the approximation to the effective action with the Kolmogorov spectrum Eq. (2.1) is qualitatively correct for 3D Navier-Stokes turbulence. Unfortunately, we have not so far been able to show that the full action, Eq. (3.18), satisfies all realizability constraints for arbitrarily large deviations δK .

It may be observed from Eq. (3.21) that the quadratic part of the approximate action has precisely the form of an Onsager-Machlup action [10]. Hence, it follows from the work of Onsager and Machlup that the same law of fluctuations is realized with the Langevin equation

$$\delta \dot{K}^+(t) + \Lambda_m p K_*^{p-1}(t) \delta K^+(t) = [2R_*(t)]^{1/2} \eta(t) \quad (3.22)$$

obtained by linearization of the energy-decay equation around its solution $K_*(t)$ and by addition of a white-noise random force $\eta(t)$, $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$, with a coefficient

$$R_*(t) = \frac{2(p-1)}{3} \varepsilon_*(t) K_*(t). \quad (3.23)$$

This alternative stochastic representation is equivalent in the sense that all finite distributions of $\delta K^+(t)$ in the above Langevin model agree with those predicted for $\delta \hat{K}(t)$ by the quadratic action. [Recall from [11,12] that $\Gamma[K]$ is a generating functional for irreducible multitime correlation functions of the energy $\hat{K}(t)$. See also Sec. III.] We emphasize that this linear Langevin representation is only adequate for the smaller fluctuations about the mean and will not be sufficient to describe the larger fluctuations. In fact, the quadratic part of the action is only a valid approximation for small deviations δK . The predicted decay of the smaller energy fluctuations according to a linearized law is similar to the Onsager regression hypothesis for equilibrium fluctuations [9]. Likewise, the expression Eq. (3.23) is a *fluctuation-dissipation relation* (FDR) analogous to that in equilibrium. The white-noise term on the right-hand side of Eq. (3.22) represents a stochastic backscatter contribution to the energy evolution and Eq. (3.23) relates its magnitude to the mean energy dissipation rate $\varepsilon_*(t)$. These are testable predictions of the closure hypotheses. We expect that the prefactor $C = \frac{2}{3}(p-1)$ in the FDR, whose precise value results from the Gaussian *Ansatz*, is correct at least on order of magnitude. For $m=2$ its value is $C = \frac{5}{9} \approx 0.56$.

The Gaussian *Ansatz* is obviously inadequate in one respect, because it fails to capture the important non-Gaussian effect of scale energy transfer. Let $\hat{\Pi}(k)$ be the usual spectral flux as an *instantaneous* variable in individual realizations, written in terms of a triple product of velocity Fourier coefficients. (For example, see [20] for an explicit expression.) Then, one expects for freely decaying turbulence in the quasi-steady regime that $\langle \hat{\Pi}(k) \rangle_{E(t)} = \varepsilon(t)$ for all inertial-range wave numbers $k_L(t) \ll k \ll k_d(t)$, breaking time-reversal symmetry. However, within the Gaussian *Ansatz* $\langle \hat{\Pi}(k) \rangle_{E(t)} = 0$. This pathology of the Gaussian *Ansatz* shows up if one calculates $\Gamma_*[\Pi]$, the Gaussian approximation to the effective action of the flux $\hat{\Pi}(k)$. In fact, $\Gamma_*[\Pi]$ is the Legendre transform of an approximate cumulant-generating functional $\lambda_*[H]$, using the notations of [12]. That is, $\Gamma_*[\Pi] = \max_H (H\Pi - \lambda_*[H])$. Because the fifth-order moment $\langle \hat{K}(\mathbf{r}) \hat{\Pi}(k) \rangle_{E(t)} = 0$ in the Gaussian *Ansatz*, as well as $\langle \hat{\Pi}(k) \rangle_{E(t)} = 0$, it follows by the methods discussed in [12] that $\lambda_*[H] \equiv 0$ for all H . Therefore the Legendre transform is

$$\Gamma_*[\Pi] = \begin{cases} 0, & \Pi = 0 \\ +\infty, & \Pi \neq 0. \end{cases} \quad (3.24)$$

This result just implies that, within the Gaussian *Ansatz*, the flux function $\hat{\Pi}(k)$ is identically zero in every realization and no fluctuations from that value may occur. This is clearly an unphysical result of the closure. However, the failure of the

Gaussian *Ansatz* to describe the energy transfer is hoped not to drastically affect the result for the effective action $\Gamma[K]$ of the energy, because the model spectrum Eq. (2.1) has built in the correct overall decay rate.

Some insight into this may be obtained by considering the *exact* equation for the two-time correlation of the energy fluctuation. The energy density fluctuation $\delta \hat{K}(\mathbf{r}, t) = \hat{K}(\mathbf{r}, t) - \langle \hat{K}(t) \rangle$ in each individual realization obeys the equation

$$\delta \dot{\hat{K}} = -\nabla \cdot [(\hat{K} + \hat{p})\mathbf{v} - \nu \nabla \hat{K}] - \delta \hat{\varepsilon}. \quad (3.25)$$

Here $\delta \hat{\varepsilon} = \hat{\varepsilon} - \langle \hat{\varepsilon} \rangle$ is the energy dissipation fluctuation and \hat{p} is the kinematic pressure. It is this equation which is being statistically modeled by the Langevin equation, Eq. (3.22). If this model is to be valid for second-order statistics, then it must be true that the exact equation

$$\begin{aligned} \langle \delta \hat{K}(t) \delta \hat{K}(t_0) \rangle &= -\langle \nabla \cdot [(\hat{K} + \hat{p})\mathbf{v} - \nu \nabla \hat{K}](t) \delta \hat{K}(t_0) \rangle \\ &\quad - \langle \delta \hat{\varepsilon}(t) \delta \hat{K}(t_0) \rangle \end{aligned} \quad (3.26)$$

coincides with the one derived from the Langevin equation. For $t > t_0$ this is just

$$\begin{aligned} \langle \delta \hat{K}^+(t) \delta \hat{K}^+(t_0) \rangle &= \langle \{-L_*(t) \delta \hat{K}^+(t) \\ &\quad + [2R_*(t)]^{1/2} \eta(t)\} \delta \hat{K}^+(t_0) \rangle \\ &= -L_*(t) \langle \delta \hat{K}^+(t) \delta \hat{K}^+(t_0) \rangle. \end{aligned} \quad (3.27)$$

We have introduced $L_*(t) = \Lambda_m p K_*^{p-1}(t)$ and noted that the white-noise force is uncorrelated with earlier values of the energy fluctuation. In order to coincide with this model equation, it is clear that the first term due to space transport on the left-hand side of Eq. (3.26) should be negligible, i.e., $\langle \nabla \cdot [(\hat{K} + \hat{p})\mathbf{v} - \nu \nabla \hat{K}](t) \delta \hat{K}(t_0) \rangle \approx 0$. This is plausible, because the space transport term is rapidly varying in time and thus decorrelated with the energy fluctuation at earlier time. It is for this reason that such higher moments (of fourth and fifth order in velocity) were never explicitly modeled in our analysis, although they are implicitly represented by the white-noise term in the Langevin equation. The remaining term in the exact equation coincides with that in the model equation, if it is further assumed that

$$\langle \delta \hat{\varepsilon}(t) | \hat{K}(s), s < t \rangle \approx -L_*(t) \delta \hat{K}(t). \quad (3.28)$$

That is, the conditional expectation of the dissipation fluctuation given the value of the energy density over the entire past should be obtained by *linearizing* the expression for mean dissipation in the closure and evaluating it at the present value of the energy fluctuation in the given realization. This conditional relation is assumed to hold when the energies $\hat{K}(t)$ are small deviations from the mean value $\langle \hat{K}(t) \rangle \approx K_*(t)$. This formula also has considerable plausibility: it is a formal statement of the ‘‘regression hypothesis’’ on small fluctuations. If this relation is used to eliminate $\delta \hat{\varepsilon}(t)$ in Eq. (3.26), then an equation of the same form as Eq. (3.27) is obtained:

$$\langle \delta \hat{K}(t) \delta \hat{K}(t_0) \rangle \approx -L_*(t) \langle \delta \hat{K}(t) \delta \hat{K}(t_0) \rangle.$$

This should make more transparent the nature of the approximations in the Rayleigh-Ritz calculation at the level of closure considered here.

In fact, it is possible by such arguments to completely ‘rederive’ the Langevin model. If one accepts (i) the hypothesis that the rapidly changing terms are correctly modeled by a white noise, i.e.,

$$-\nabla \cdot [(\hat{K} + \hat{p})\mathbf{v} - \nu \nabla \hat{K}](t) - [\delta \hat{\varepsilon}(t) - \langle \delta \hat{\varepsilon}(t) | \hat{K}(s), s < t \rangle] \approx [2R_*(t)]^{1/2} \eta(t), \quad (3.29)$$

and (ii) the ‘‘regression hypothesis’’ in Eq. (3.28), then the exact equation (3.25) reduces to the Langevin model Eq. (3.22). A Kolmogorov-style dimensional analysis would yield for the noise strength $R_*(t) = C \varepsilon_*(t) K_*(t)$ with C some universal constant, to be determined. The value of this constant $C = \frac{2}{3}(p-1)$ resulting from the variational calculation with the Gaussian *Ansatz* will furthermore be shown below to be the unique choice to recover the relation Eq. (4.5). Since this relation is exact when single-point statistics of velocity are Gaussian—which is known to be a very good approximation—the Langevin model can be entirely motivated by intuitive considerations. The variational calculation is a systematic analytical procedure yielding the same Langevin model, but also only approximate. The two derivations are therefore very complementary.

Improvements of the Gaussian *Ansatz* are likely to lead to better results for the effective actions. For example, the ‘‘synthetic turbulence’’ models of [21] are random velocity fields which contain the correct energy transfer and also some of the intermittency effects of real turbulent velocities. Using such statistical models within our Rayleigh-Ritz scheme should lead not only to a qualitatively correct result for $\Gamma[\Pi]$ but also to quantitatively better results for $\Gamma[K]$. In such improved closures new ‘‘test functionals’’ in addition to the quadratic moment functional $\hat{K}(\mathbf{r}; \mathbf{v})$ must be considered to determine the time dependence of the additional free parameters in the statistical *Ansatz*. For example, the energy flux variable (a triple moment functional) would be a natural variable to add to the closure. The choice of the ‘‘test functionals’’ is an equally important element of the closure as is the choice of the model velocity statistics. We emphasize again that our results above for $\Gamma[K]$ depend very little on the choice of the Gaussian statistics. For *any* model statistics with the mean energy $E(t)$ as the only free parameter and with $\hat{K}(\mathbf{r}; \mathbf{v})$ the corresponding ‘‘test functional,’’ results very similar to those above will follow. In that general setting a result $\langle \hat{K}^2(\mathbf{r}) \rangle_{E(t)} = B E^2(t)$ will hold by dimensional analysis, for some constant B , replacing Eq. (3.14). This means that the formula Eq. (3.18) for the approximate action will still hold, with the factor $\frac{3}{2}$ in front simply replaced by another number $D = 1/(B-1)$ of order unity. Only the value $B = \frac{5}{3}$ depends upon the Gaussian *Ansatz*. By employing improved closures one may hope to derive from first principles such theoretical features as the ‘‘permanence of large eddies’’ for $m < 4$. Because the Rayleigh-Ritz algorithm is a convergent approximation scheme for the true effective ac-

tions, systematic improvement of the closures will lead to a refined description of the turbulent dynamics.

IV. TESTING THE THEORY

The previous theory has testable consequences for turbulent energy fluctuations. The most likely experimental situation for such checks is grid turbulence, which well approximates a homogeneous, isotropic, decaying turbulence. In principle, it would be possible to make an experiment by measuring the velocity at a single point \mathbf{r} in the frame of mean downstream motion. Constructing from this the energy history $\hat{K}(t) \equiv \frac{1}{2} v^2(\mathbf{r}, t)$ and compiling an ensemble of realizations, one may, as in [17,22], compute various multitime statistics to compare with the theory.

The most familiar such statistics are the r -time correlation functions $\langle \hat{K}(t_1) \cdots \hat{K}(t_r) \rangle$. These may be derived directly from the effective action $\Gamma[K]$. In fact, by taking r functional derivatives of the action, evaluated at the mean value, the *irreducible* r -time correlators are obtained:

$$\langle \hat{K}(t_1) \cdots \hat{K}(t_r) \rangle^{\text{irr}} = \frac{\delta^r \Gamma[K]}{\delta K(t_1) \cdots \delta K(t_r)} \Big|_{K=K_*}. \quad (4.1)$$

For this result, for the definition of irreducible correlators and their relation to the connected correlators (or cumulants), see any text in quantum field theory, e.g., [23], Sec. 6.2.2 or [24], Sec. 10.2. We only note here that the irreducible two-time correlator, or $\langle \hat{K}(t_1) \hat{K}(t_2) \rangle^{\text{irr}}$, is the inverse operator kernel of the connected two-time function (covariance) $\langle \hat{K}(t_1) \hat{K}(t_2) \rangle^{\text{con}} = \langle \delta \hat{K}(t_1) \delta \hat{K}(t_2) \rangle$, i.e.,

$$\int ds \langle \hat{K}(t) \hat{K}(s) \rangle^{\text{irr}} \langle \delta \hat{K}(s) \delta \hat{K}(t') \rangle = \delta(t-t'). \quad (4.2)$$

Relations between higher-order irreducible and connected correlators are obtained by taking further functional derivatives of this relation with respect to K . See [23,24].

It is very easy to obtain the variational approximation $\langle \hat{K}(t) \hat{K}(s) \rangle_*^{\text{irr}}$ from Eq. (4.1) and the quadratic part of the Gaussian effective action, Eq. (3.21). Taking its inverse, the covariance $\langle \delta \hat{K}(t) \delta \hat{K}(t') \rangle_*$ is then evaluated as

$$\begin{aligned} \langle \delta \hat{K}(t) \delta \hat{K}(t') \rangle_* &= \exp \left[- \int_{t_0}^t ds L_*(s) - \int_{t_0}^{t'} ds L_*(s) \right] \\ &\quad \times (\delta K_0)^2 + 2 \int_{t_0}^{\min\{t,t'\}} ds R_*(s) \\ &\quad \times \exp \left[- \int_s^t dr L_*(r) - \int_s^{t'} dr L_*(r) \right] \end{aligned} \quad (4.3)$$

in our theory. We have again written $L_*(t) = \Lambda_m p K_*^{p-1}(t)$. These calculations are outlined in Appendix B. In the same way, by taking an arbitrary number r of functional derivatives in Eq. (4.1), all correlations of any finite order are obtainable from the effective action. However, we do not pursue the general calculation here.

To cast the theoretical results into a form that may be compared with experiment, we insert the mean decay law $K_*(t)$ from Eq. (2.5) into Eq. (4.3) and perform the integrals. For the two-time covariance of the turbulent energy this calculation is straightforward and the prediction is

$$\begin{aligned} \langle \delta \hat{K}(t) \delta \hat{K}(t') \rangle_* &= \left(\frac{t-t_0^*}{\Delta t} \right)^{-(n+1)} \left(\frac{t'-t_0^*}{\Delta t} \right)^{-(n+1)} \\ &\times \left\{ (\delta K_0)^2 + \frac{2}{3} K_0^2 \left[\left(\frac{t_{\min}-t_0^*}{\Delta t} \right)^2 - 1 \right] \right\}, \end{aligned} \quad (4.4)$$

with $t_{\min} = \min\{t, t'\}$. The notations are the same as for the mean decay law. It should be noted that the first term $\propto (\delta K_0)^2$ corresponds to decay of an initial energy fluctuation δK_0 . The second term $\propto K_0^2$ represents the new fluctuations generated by the internal turbulence noise, through the stochastic backscatter dynamics. As a consequence of that term, the long-time rms value of the energy, $K_{\text{rms}}(t) = \{ \langle [\delta \hat{K}(t)]^2 \rangle_* \}^{1/2}$, evolves to a constant level with respect to the mean energy $K_*(t)$:

$$\lim_{t \rightarrow \infty} \frac{\langle [\delta \hat{K}(t)]^2 \rangle_*}{K_*^2(t)} = \frac{2}{3}. \quad (4.5)$$

The limiting value of $\frac{2}{3}$ is what would occur for an asymptotic Gaussian statistics of the one-point velocity variable. If we had adopted a non-Gaussian *Ansatz* in our calculation, then the predicted limiting value would have been $1/D = (B-1)$. Since any value of the constant can be accommodated by an appropriate such *Ansatz*, it is not so important to the theory which particular constant is correct (although the Gaussian value is expected to be quite accurate). What would falsify the present theory would be an experimental finding that the functional form in Eq. (4.4) was wrong, for any possible choice of the constant $1/D$ replacing $\frac{2}{3}$. We should emphasize, however, that the standard theory for the mean energy decay law $K_*(t)$ in Eq. (2.5) is in agreement with present experiments, with a value of n near 1.2. In all cases studied so far, PDF *Ansätze* adequate to predict the mean values of selected variables have also, employed in our Rayleigh-Ritz method, yielded good predictions for the fluctuations of those variables near the means. See [8]. We therefore expect the prediction in Eq. (4.4) to be reasonably accurate.

The previous result for $r=2$, the covariance in Eq. (4.3), may be obtained as well from the Langevin equation, Eq. (3.22). However, it must be emphasized that only the *small* fluctuations of the energy, with $\delta \hat{K}(t) \ll K_*(t)$, are expected to be distributed according to that linearized equation. Because correlation functions will get sizable contributions from the larger fluctuations, for which the linear law breaks down, it would not be appropriate to compare general r -time correlation functions of $\hat{K}(t)$ obtained from the linear theory with experiment. It only happens for $r=2$ that the linear Langevin equation and the full (nonlinear) effective action yield the same predictions.

It is possible to give both the effective action functional and the linear Langevin equation a direct empirical signifi-

cance in grid turbulence. This is based upon the standard device of making N independent trials to calculate the averages from experiment. Indeed, performing the same decay experiment N times independently, one usually considers an *empirical mean history*

$$\bar{K}_N(t) = \frac{1}{N} \sum_{i=1}^N \hat{K}_i(t) \quad (4.6)$$

formed from the realizations $\hat{K}_i(t)$, $i=1, 2, \dots, N$, of the N different samples. (Assuming that the turbulence is indeed statistically homogeneous, these N measurements might even be taken from points \mathbf{r}_i , $i=1, \dots, N$ in the same flow but at separations greater than L , the integral scale, to assure statistical independence.) The effective action $\Gamma[K]$ measures the probability for the empirical mean $\bar{K}_N(t)$ (which is a random quantity at finite N) to take on a value very different from the true ensemble average $\langle \hat{K}(t) \rangle$. More precisely,

$$\text{Prob}\{[\bar{K}_N(t) \approx K(t) : t_0 \leq t < +\infty]\} \sim \exp(-N\Gamma[K]). \quad (4.7)$$

Thus the probability to observe $\bar{K}_N(t)$ taking any value $K(t)$ other than the true ensemble average $\langle \hat{K}(t) \rangle$ is exponentially small in the number of samples N . This is a consequence of the famous Cramér theorem on large deviations of sums of independent random variables (e.g., see [7], Sec. 8.6.4, and references therein). Put another way, the dimensionless quantity $1/\Gamma[K]$ gives an estimate of the number N of additional independent samples required to reduce by e -fold the probability of the fluctuation value K in the empirical average \bar{K}_N . In principle, therefore, the effective action $\Gamma[K]$ is itself directly measurable in grid turbulence, by assembling a histogram of observed histories $\bar{K}_N(t)$ and determining the decay rate of the probabilities for large N . However, this would not be feasible with a reasonable number of independent samples N except for histories $K(t)$ sufficiently near the mean history.

The linear Langevin model Eq. (3.22) is more restricted in its validity, since, as has been stressed, it is equivalent to the quadratic action and is adequate only to predict second-order statistics of $\hat{K}(t)$. In particular, a linear Langevin equation can produce only Gaussian multitime statistics for $\delta K^+(t)$. (Of course, this has nothing to do with the use of a Gaussian PDF *Ansatz* for the *velocity field* and will be true even if a non-Gaussian *Ansatz* is employed: see Appendix A.) The true statistics of $\delta \hat{K}(t)$ will not be Gaussian at all, e.g., they will be χ^2 if the one-point velocity itself is Gaussian. Nevertheless, the Langevin equation can also be given a direct empirical significance in terms of the independent N -sample ensemble, based upon the central limit theorem. It accurately describes the statistics of the normalized sum variable:

$$\delta \hat{K}_N(t) = \frac{1}{N^{1/2}} \sum_{i=1}^N [\hat{K}_i(t) - \langle \hat{K}(t) \rangle]. \quad (4.8)$$

This quantity has zero mean and the same covariance as $\delta \hat{K}_i(t) = \hat{K}_i(t) - \langle \hat{K}_i(t) \rangle$ for each independent sample i , i.e., $\langle \delta \hat{K}_N(t) \delta \hat{K}_N(t') \rangle = \langle \delta \hat{K}_i(t) \delta \hat{K}_i(t') \rangle$ for all i . However, it is furthermore a Gaussian variable in the limit of large N , in

agreement with the linear Langevin equation. Thus, at a large but finite N it is legitimate to compare predictions of the correlations of $\delta K^+(t)$ using the linear Langevin dynamics, Eq. (3.22), with those from experiment for $\delta \hat{K}_N(t)$ at large N . Notice that the Gaussian statistics for $\delta \hat{K}_N(t)$ in fact result by substituting into $\Gamma[K]$ in Eq. (4.7) the value $K = \langle \hat{K} \rangle + N^{-1/2} \delta K$. In that case, by expanding in δK , one obtains

$$\text{Prob}[\{\delta \hat{K}_N(t) \approx \delta K(t) : t_0 \leq t < +\infty\}] \sim \exp(-\Gamma_{(2)}[\delta K]), \quad (4.9)$$

where $\Gamma_{(2)}[\delta K]$ is the quadratic approximation to the exact effective action and terms in the exponent of order $N^{-1/2}$ have been neglected in the large N limit. This is just one of the standard proofs of the central limit theorem. The important point here is that it naturally explains why the linear Langevin equation may be appropriate to calculate second-order statistics but certainly not higher order. For the latter purpose the full nonlinear action, Eq. (3.18), must be used, not just the quadratic part.

V. CONCLUSIONS

The main results of this work are as follows.

(1) We have derived an action functional, Eq. (3.18), for energy histories in decaying homogeneous and isotropic turbulence at high Reynolds number, by means of a Rayleigh-Ritz calculation using standard closure assumptions. This action generalizes the Onsager-Machlup action to fully developed turbulent flow and characterizes the mean energy history by a variational principle.

(2) We have shown that the quadratic part of the action, Eq. (3.21), valid for a region of small fluctuations sufficiently near the mean, satisfies all required realizability constraints. In fact, it is of the Onsager-Machlup form and thus has a stochastic realization by a linear Langevin equation for

the energy history. The deterministic part of the equation is obtained by linearization of the mean decay law and the random part has its strength determined by a ‘‘fluctuation-dissipation relation’’ in terms of the mean energy dissipation.

(3) Testable consequences of the theory are r -time correlation functions of the energy, which may be obtained by functional differentiation of the effective action. These are not obtainable from the starting hypotheses by direct PDF methods. As an example, the two-time cumulant, or covariance, of the energy history is derived in detail.

(4) A direct empirical significance of the effective action is given in terms of fluctuation probabilities for ensemble averages over N independent samples or ensemble points. This interpretation permits the effective action itself to be measured experimentally, at least for arguments in path space sufficiently close to the mean history.

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APPENDIX A: QUADRATIC APPROXIMATION TO THE ACTION

Let us consider in Eq. (3.18) small fluctuations $\delta K, \delta E$:

$$K(t) = K_*(t) + \delta K(t) \quad \text{and} \quad E(t) = K_*(t) + \delta E(t), \quad (A1)$$

using the fact that $E_*(t) = K_*(t)$. It is then easy to see from Eq. (3.18) that, up to quadratic order,

$$\Gamma_*[K] = \frac{3}{2(p-2)\Lambda_m} \int_{t_0}^{\infty} dt \frac{[\delta \dot{K}(t) + \Lambda_m p K_*^{p-1}(t) \delta E(t)][\delta \dot{E}(t) + \Lambda_m p K_*^{p-1}(t) \delta E(t)]}{K_*^{p+1}(t)}. \quad (A2)$$

By a straightforward linearization of the determining Eq. (3.17), it follows that

$$-2(p-1)\Lambda_m K_*^{p-1}(\delta E - \delta K) = \delta \dot{K} + \Lambda_m p K_*^{p-1} \delta K. \quad (A3)$$

Let us introduce a shorthand notation for the left-hand side of this equation:

$$\Delta \equiv \delta \dot{K} + \Lambda_m p K_*^{p-1} \delta K. \quad (A4)$$

Using Eq. (A3), it is not hard to show that

$$\delta \dot{K}(t) + \Lambda_m p K_*^{p-1}(t) \delta E(t) = \frac{(p-2)}{2(p-1)} \Delta \quad (A5)$$

and

$$\begin{aligned} & \delta \dot{E}(t) + \Lambda_m p K_*^{p-1}(t) \delta E(t) \\ &= \frac{1}{2} \Delta - K_* \frac{d}{dt} \left(\frac{\Delta}{2(p-1)\Lambda_m K_*^p} \right). \end{aligned} \quad (A6)$$

Substituting these into Eq. (A2), it follows that, to quadratic order,

$$\begin{aligned}
\Gamma_*[K] &= \frac{3}{2(p-2)\Lambda_m} \int_{t_0}^{\infty} dt \frac{(p-2)}{2(p-1)} \frac{\Delta}{K_*^{p+1}} \\
&\quad \times \left[\frac{1}{2} \Delta - K_* \frac{d}{dt} \left(\frac{\Delta}{2(p-1)\Lambda_m K_*^p} \right) \right] \\
&= \frac{3}{8(p-1)\Lambda_m} \int_{t_0}^{\infty} dt \frac{\Delta^2(t)}{K_*^{p+1}(t)} \\
&\quad - \frac{3}{16(p-1)^2 \Lambda_m^2} \int_{t_0}^{\infty} dt \frac{d}{dt} \left[\left(\frac{\Delta}{K_*^p} \right)^2 \right] \\
&= \frac{3}{8(p-1)\Lambda_m} \int_{t_0}^{\infty} dt \frac{\Delta^2(t)}{K_*^{p+1}(t)}. \quad (\text{A7})
\end{aligned}$$

To obtain the last line we used the boundary conditions $\Delta(t_0) = \Delta(\infty) = 0$, which are required by Eqs. (3.4), (3.5), and (3.15). It should be obvious that the last line of Eq. (A7) is the same as the result, Eq. (3.21), claimed in the text.

Although the present calculation employed the Gaussian *Ansatz*, it should be stressed that a similar result will hold for more realistic statistical models of the velocity field. In particular, the quadratic form of the action does not depend upon the Gaussian assumption, but is simply a consequence of the fact that the mean history $K_*(t)$ is required to be an absolute minimum. Hence, this will be true for any closure

model leading to an effective action satisfying the realizability conditions. In that case, an expansion in small deviations $\delta K(t)$ around the mean must necessarily lead to a quadratic expression involving the linearized evolution expression, $\Delta(t) = \delta \dot{K} + \Lambda_m p K_*^{p-1} \delta K$. What will be different for other closures is the coefficient multiplying $\Delta^2(t)$, which correspond to different predictions of the fluctuations around the mean.

APPENDIX B: THE PREDICTED TWO-TIME CUMULANT

As observed in the text, the irreducible r -time correlations of $\hat{K}(t)$ can be obtained from functional derivatives of $\Gamma[K]$ at $K = K_*$: see Eq. (4.1). Equivalently, these irreducible correlators can be read off from the Taylor series:

$$\begin{aligned}
\Gamma[K] &= \sum_{r=2}^{\infty} \frac{1}{r!} \int dt_1 \cdots \int dt_r \Gamma_r(t_1, \dots, t_r) \\
&\quad \times \delta K(t_1) \cdots \delta K(t_r), \quad (\text{B1})
\end{aligned}$$

as $\langle \hat{K}(t_1) \cdots \hat{K}(t_r) \rangle^{\text{irr}} = \Gamma_r(t_1, \dots, t_r)$. It is thus easy to obtain in the Gaussian approximation $\langle \hat{K}(t_1) \hat{K}(t_2) \rangle_*^{\text{irr}}$ from the quadratic part of the Gaussian action derived in Appendix A, written as

$$\Gamma_*[K] = \frac{1}{2} \int dt_1 \int dt_2 \frac{[\delta \dot{K}(t_1) + L_*(t_1) \delta K(t_1)][\delta \dot{K}(t_2) + L_*(t_2) \delta K(t_2)]}{2R_*(t_1)} \delta(t_1 - t_2). \quad (\text{B2})$$

Then,

$$\begin{aligned}
\langle \hat{K}(t_1) \hat{K}(t_2) \rangle_*^{\text{irr}} &= [-\partial_{t_1} + L_*(t_1)][2R_*(t_1)]^{-1} \\
&\quad \times [\partial_{t_1} + L_*(t_1)] \delta(t_1 - t_2) \quad (\text{B3})
\end{aligned}$$

and, from Eq. (4.2), the two-time cumulant must satisfy

$$\begin{aligned}
[-\partial_{t_1} + L_*(t_1)][2R_*(t_1)]^{-1} [\partial_{t_1} + L_*(t_1)] \\
\times \langle \delta \hat{K}(t_1) \delta \hat{K}(t_2) \rangle_* = \delta(t_1 - t_2). \quad (\text{B4})
\end{aligned}$$

To solve this equation, we use the Green functions

$$[-\partial + L_*]^{-1}(t, t') = \exp \left[\int_{t'}^t ds L_*(s) \right] \theta(t' - t), \quad (\text{B5})$$

$$[\partial + L_*]^{-1}(t, t') = \exp \left[- \int_{t'}^t ds L_*(s) \right] \theta(t - t'), \quad (\text{B6})$$

which are anticausal and causal, respectively. Performing one integration we obtain, for $s \leq t_2$,

$$\begin{aligned}
[\partial_s + L_*(s)] \langle \delta \hat{K}(s) \delta \hat{K}(t_2) \rangle_* \\
= 2R_*(s) [-\partial_s + L_*(s)]^{-1} \delta(s - t_2) \\
= 2R_*(s) \exp \left[- \int_s^{t_2} dr L_*(r) \right] \theta(t_2 - s) \\
\equiv G(s; t_2). \quad (\text{B7})
\end{aligned}$$

A second integration forward from t_0 gives, for $t_1 \leq t_2$,

$$\begin{aligned}
\langle \delta \hat{K}(t_1) \delta \hat{K}(t_2) \rangle_* = \exp \left[- \int_{t_0}^{t_1} ds L_*(s) \right] F(t_2) \\
+ \int_{t_0}^{t_1} ds \exp \left[- \int_s^{t_1} dr L_*(r) \right] G(s; t_2). \quad (\text{B8})
\end{aligned}$$

The first term on the right is an arbitrary solution of the homogeneous equation. The function $F(t_2)$ is determined by symmetry and initial conditions as

$$F(t_2) = \exp \left[- \int_{t_0}^{t_2} ds L_*(s) \right] \langle [\delta \hat{K}(t_0)]^2 \rangle. \quad (\text{B9})$$

When this is substituted into Eq. (B8) along with the definition of $G(s; t_2)$ from Eq. (B7), the claimed result, Eq. (4.3)

in the text, is obtained. Note that $\langle [\delta \hat{K}(t_0)]^2 \rangle$ defines the quantity $(\delta K_0)^2$ in Eq. (4.3) of the text.

It is easy to check that this result coincides also with $\langle \delta K^+(t) \delta K^+(t_0) \rangle$ calculated from the Langevin model, Eq. (3.22). However, in general, for correlations of order $r > 2$, the Langevin model is not adequate and the above method of calculation from the effective action must be employed.

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